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Ultra-relativistic geometrical shock dynamics and vorticity

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Geometrical shock dynamics, also called CCW theory, yields approximate equations for shock propagation in which only the conditions at the shock appear explicitly; the post-shock flow is presumed approximately uniform and enters implicitly via a Riemann invariant. The non-relativistic theory, formulated by G. B. Whitham and others, matches many experimental results surprisingly well. Motivated by astrophysical applications, we adapt the theory to ultra-relativistic shocks advancing into an ideal fluid whose pressure is negligible ahead of the shock, but is one third of its proper energy density behind the shock. Exact results are recovered for some self-similar cylindrical and spherical shocks with power-law pre-shock density profiles. Comparison is made with numerical solutions of the full hydrodynamic equations. We review relativistic vorticity and circulation. In an ultra-relativistic ideal fluid, circulation can be defined so that it changes only at shocks, notwithstanding entropy gradients in smooth parts of the flow.

1. Introduction

Gamma-ray-burst afterglows have spurred us to consider this problem. These cosmologically distant events are believed to involve a shock launched by the death of a massive star with initial Lorentz factor $\Gamma_0 > 10^2$ relative to a pre-shock circumstellar wind (mass density $\rho_0 \propto r^{-2}$) or interstellar medium ($\rho_0 \sim$ constant): see van Paradijs, Kouveliotou & Wijers (2000), Piran (2005) and Mészáros (2006) for reviews. Light curves fluctuate strongly at early times, probably because of unsteadiness in the source; later, brightness falls approximately as a power law in time but often with undulations that may be due to inhomogeneities ahead of the shock. The observed radiation appears to be synchrotron emission, which implies that $\gtrsim 10^{-2}$ of the post-shock energy density takes the form of magnetic field and highly relativistic electrons. Even after compression by the shock, typical circumstellar or interstellar fields would be many orders of magnitude too small. Therefore, it is often supposed that magnetic energy is created rapidly by plasma instabilities at the shock front (Medvedev & Loeb 1999). We wish to explore whether the compressed pre-shock field might instead be amplified gradually by macroscopic fluid turbulence. The source of the turbulence is supposed to be vorticity produced as the shock passes over inhomogeneities in the ambient medium. Astrophysical applications, however, are discussed elsewhere (Sironi & Goodman 2007). Our purpose here is to develop

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and test suitable theoretical tools: a relativistic version of geometric shock dynamics (hereafter GSD); and, independently, a suitable redefinition of relativistic vorticity that leads to conservation of the circulation on any fluid contour that does not cross a shock.

The elements of non-relativistic GSD were developed in the 1950s (Moeckel 1952; Chester 1954; Chisnell 1957; Whitham 1957, 1958, 1959). Whitham (1974) gives a pedagogical review, upon which we have relied heavily. The one-dimensional version of the theory gives a functional or even algebraic relationship (rather than a partial differential equation) between variations in the pre-shock density and variations in the shock Mach number. The multidimensional version describes the effect of changes in shock area – divergence or convergence of the shock normals – on the Mach number. Thus, the theory reduces the dimensionality of the problem by one: in three dimensions, for example, it gives a closed set of equations for the evolution of the shock surface. GSD has even been adapted to reacting flows (detonation waves: Li & Ben-Dor 1998).

Naturally, there is a price to be paid in accuracy for these simplifications. For a recent critique, see Baskar & Prasad (2005). Nevertheless, GSD often performs remarkably well when there is reason to expect that fluid gradients or geometrical constraints near the shock should dominate, rather than reflections from boundaries behind the shock, and even in some cases where there is no such expectation. GSD successfully describes diffraction of shocks around corners, acceleration of converging shocks, and even the propagation of kinks (‘shock shocks’) along shock fronts, as judged by comparisons with experiment and with exact self-similar solutions (Bryson & Gross 1961; Schwendeman 1988; Whitham 1974, and references therein).

Relativistic units in which the speed of light $c = 1$ will be used. We adopt the conventions of Schutz (1990) for tensors; in particular, the metric in Minkowski coordinates $x^{\mu} = (x^0 = t, x^1, x^2, x^3)$ is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, while $T^{\mu\nu}$ and $T^{\bar{\mu}\bar{\nu}}$ denote the components of the same tensor in two Lorentz frames $\mathscr{O}$, $\bar{\mathscr{O}}$. In all cases considered here, the energy density of the pre-shock fluid will be dominated by rest mass, so that pressure and turbulent motions can be neglected ahead of the shock.

2. Planar shocks

Our goal is to transcribe GSD for an ultra-relativistic ideal fluid. Following Whitham (1974), we begin with the case that the area of the shock is constant and the pre-shock density (ρ_0) is stratified on planes parallel to the shock front. In place of Mach number, we will be concerned with shock Lorentz factor ($\varGamma$) or rapidity parameter ($\varPhi$), the two being related by $\varGamma \equiv \cosh\varPhi \approx \mathrm{e}^{\varPhi}/2 \gg 1$. These quantities are defined in the rest-frame of the pre-shock medium.

The construction of GSD proceeds in two parts. First, the jump conditions are derived from the basic conservation laws; these relate the post-shock fluid properties to the pre-shock ones if $\varGamma$ is given. This step is potentially exact but simplifies after approximations based on $\varGamma \gg 1$, $\rho \approx 3P$, and $P_0 \ll \rho_0$. Next, characteristic equations are derived for the post-shock flow, and the (uncontrolled) approximation is made that one of the Riemann invariants has a known and uniform value behind the shock.

2.1. *Jump conditions*

For a planar shock propagating in the x^1-direction, the relevant components of the energy-momentum tensor are

$$T^{00} = (\rho + P)\gamma^2 - P, \quad T^{01} = (\rho + P)\gamma^2\beta, \quad T^{11} = (\rho + P)\gamma^2\beta^2 + P, \tag{2.1}$$

where the fluid 4-velocity has components

$$U^{\mu} \rightarrow (\gamma, \gamma\beta, 0, 0) \equiv (\cosh\phi, \sinh\phi, 0, 0)$$

measured in the shock rest frame. Here $\boldsymbol{\beta} = \boldsymbol{v}/c$ is the conventional velocity (i.e. 3-velocity) of the flow relative to the speed of light, $\gamma = (1 - \beta^2)^{-1/2}$ is the corresponding Lorentz factor, and the rapidity parameter ϕ is defined so that $\beta = \tanh\phi$. The latter is convenient when all relative motions of interest are parallel to a single axis (the x-axis in the present case), because it simplifies the addition of velocities: that is, if the velocities of A relative to B and of B relative to C are $\beta_{AB} = \tanh\phi_{AB}$ and $\beta_{BC} = \tanh\phi_{BC}$, respectively, then $\beta_{AC} = (\beta_{AB} + \beta_{BC})/(1 + \beta_{AB}\beta_{BC})$, whereas $\phi_{AC} = \phi_{AB} + \phi_{BC}$ (Taylor & Wheeler 1966). The proper energy density ρ and pressure P are defined in the local fluid rest frame, so that they are Lorentz invariants.

The jump conditions in the shock rest frame are that T^{01} and T^{11} should be continuous. Since $P \approx \rho/3$ behind the shock and $P_0 \ll \rho_0$ in front of it, these conditions become

$$T^{01} : \ 4P\gamma^2\beta = -\rho_0\,\Gamma^2 \qquad T^{11} : \ 4P\gamma^2\beta^2 + P = \rho_0\,\Gamma^2. \tag{2.2}$$

Consistent with the ultra-relativistic approximation, the pre-shock 3-velocity has been set to -1, which incurs an error $\sim O(\Gamma^{-2})$. Eliminating $\Gamma^2\rho_0$ between the two equations, dividing through by P, and setting $\gamma^{-2} \rightarrow 1-\beta^2$ yields $(3\beta+1)(\beta+1) = 0$. The root $\beta = -1$ corresponds to no shock at all. Therefore, $\beta = -1/3$ in the shock frame. In terms of the rapidity parameters of the fluid and the shock, $\tanh(\Phi - \phi) = 1/3$, which is a covariant formulation since a Lorentz boost along x^1 with velocity v simply adds $\tanh^{-1} v$ to both ϕ and Φ. Substituting $\beta = -1/3$ and $\gamma^2 = 9/8$ into either of (2.2) yields $P = (2/3)\Gamma^2\rho_0$. So, the jump conditions are

$$\phi = \Phi - \tanh^{-1}\tfrac{1}{3} = \Phi - \ln\sqrt{2}\,, \tag{2.3a}$$

$$\zeta \equiv \frac{\sqrt{3}}{4}\ln P \approx \frac{\sqrt{3}}{4}\left(\ln\rho_0 + 2\Phi - \ln 6\right)\,. \tag{2.3b}$$

The peculiar factor $\sqrt{3}/4$ will simplify the characteristic equations below.

2.2. *Whitham's characteristic rule*

Equations (2.3) give two relations among the four variables $(\phi, \Phi, \zeta, \ln\rho_0)$, or equivalently, $(\gamma, \Gamma, P, \rho_0)$. They are exact up to terms $O(\Gamma^{-2})$. Whitham's formulation of GSD adds one more condition: the Riemann invariant associated with the characteristics that go upstream from the post-shock flow toward the shock, R_+, is supposed to have the same value as it would if the shock were a transition between constant states. The rationale is that the perturbations to the shock front are supposed to be localized; they are driven by small-scale density variations in the pre-shock fluid, or by local geometrical constraints on the shock, and therefore these perturbations are supposed to average out downstream.

So the next step is to derive the Riemann characteristics from the equations of motion $T^{\mu\nu}{}_{,\nu} = 0$. This has been done in greater generality elsewhere (Johnson & McKee 1971; Martí & Müller 1994), but for completeness we shall rederive the special case we need. With (2.1), the equations of motion become

$$T^{0\nu}{}_{,\nu} \propto (2\cosh 2\phi + 1)\frac{4}{\sqrt{3}}\dot{\zeta} + (4\sinh 2\phi)\dot{\phi} + (2\sinh 2\phi)\frac{4}{\sqrt{3}}\zeta' + (4\cosh 2\phi)\phi' = 0,$$

$$T^{1\nu}{}_{,\nu} \propto (2\sinh 2\phi)\frac{4}{\sqrt{3}}\dot{\zeta} + (4\cosh 2\phi)\dot{\phi} + (2\cosh 2\phi - 1)\frac{4}{\sqrt{3}}\zeta' + (4\sinh 2\phi)\phi' = 0,$$

in which the overdots denote $\partial/\partial x^0$ and the primes $\partial/\partial x^1$. Rather than manipulate these equations directly, it is easier to boost into the local rest frame where $\phi = 0$, find the characteristics there, and then boost back. Since $\cosh 2\phi \to 1$ and $\sinh 2\phi \to 0$, the equations above reduce to

$$\dot{\zeta} + \frac{1}{\sqrt{3}}\phi' = 0, \qquad \dot{\phi} + \frac{1}{\sqrt{3}}\zeta' = 0.$$

By adding and subtracting these, one sees that the characteristic velocities are $\pm 1/\sqrt{3}$, and the corresponding invariants $\zeta \pm \phi$. Boosting along x^1 to any other frame simply adds a constant to ϕ. Therefore,

$$R_+ \equiv \zeta + \phi \text{ is constant on } C_+ : \quad \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_+ = \tanh\left(\phi + \tanh^{-1}\frac{1}{\sqrt{3}}\right), \qquad (2.4a)$$

$$R_- \equiv \zeta - \phi \text{ is constant on } C_- : \quad \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_- = \tanh\left(\phi - \tanh^{-1}\frac{1}{\sqrt{3}}\right). \qquad (2.4b)$$

In Whitham's approximation, R_+ is constant not only along the C_+ characteristics but everywhere in the post-shock flow, even immediately behind the shock. Therefore, evaluating ϕ and ζ from the jump conditions (2.3), one obtains

$$\Phi + \lambda \ln \rho_0 \approx \text{constant}, \qquad \text{where } \lambda \equiv \sqrt{3} - \tfrac{3}{2} \approx 0.232. \qquad (2.5)$$

This approximate equation predicts how the shock speed slows in response to a transitory increase in pre-shock density: $\Gamma \propto \rho_0^{-\lambda}$. With (2.3), we obtain the corresponding changes in post-shock rapidity and pressure:

$$\phi + \lambda \ln \rho_0 = \text{constant}; \qquad P \propto \rho_0^{1-2\lambda}. \qquad (2.6)$$

A terse but rather general discussion of planar ultra-relativistic shocks, as well as expansion into vacuum, is given by Johnson & McKee (1971). The exponent we call λ is denoted s in their paper. Like us, they take the pre-shock medium to have negligible internal motions and pressure. They restrict themselves to cases in which the pre-shock density declines with increasing distance in front of the shock. This ensures that the Riemann invariants are strictly conserved behind the shock. We however allow for increases as well as decreases in ρ_0; this can give rise to reverse shocks, but while the Riemann invariants are then not strictly conserved, we will show that they are very resilient, so that (2.6) remains a good approximation for very large and abrupt changes in ρ_0 provided only that the final Lorentz factor remains $\gg 1$.

2.3. *Comparison with an exact self-similar solution*

To reiterate, the approximation (2.5) is intended to describe localized and transitory fluctuations in the propagation of a shock that has some prescribed average Lorentz factor $\bar{\Gamma}$ and advances into a 'cold' medium with some prescribed, but spatially variable, pre-shock density ρ_0 and negligible internal motions and pressure. Whitham (1974) shows that the original non-relativistic version of his theory approximates rather well the self-similar propagation of a planar shock from $x < 0$ into a power-law density profile $\rho \propto (-x)^n$, even though this situation does not entirely satisfy the assumptions of GSD.

The corresponding ultra-relativistic solution is discussed by Sari (2006). The special case of an exponentially declining pre-shock density, treated in passing by Johnson & McKee (1971), has been discussed in more detail by Perna & Vietri (2002). In Sari's terminology, the case of interest at the moment is a planar (dimensionality parameter $\alpha = 0$) Type II solution: a 'diverging' shock expanding into a pre-shock density $\propto x^{-k}$ for $0 < x < \infty$ with $k > 2$, or a 'converging' solution advancing from $x < 0$ into $\rho_0 \propto (-x)^{-k}$ with $k < 1 + \sqrt{3/4}$. The Type I (II) solutions are those in which the power-law scaling of shock position with time can (cannot) be deduced from global energy conservation (Barenblatt 1996). Type II, where the scalings are determined by local conditions near the shock – rather than the inertia of the 'piston' behind it – is the case for which one might hope that Whitham's theory would have some success. Indeed, Sari's equation (26) shows that the shock evolves as $\Gamma \propto \rho^{-\lambda}$ with λ exactly as in (2.5). The exactness of the agreement is surprising, for while Johnson & McKee (1971) showed that (2.6) holds exactly for a shock expanding down a decreasing density gradient, they assumed an initially uniform Riemann invariant R_+. The explanation probably has to do with the fact that in the Type II solutions, a sonic point separates the immediate post-shock flow from regions farther downstream; the shock outruns the possibly non-uniform conditions there. Equation (2.6) does not agree with the Type I solutions, however: for example, for a diverging shock with $k = 0$ (a planar shock resulting from an explosion in a uniform medium bounded by a stationary rigid wall), Sari's equation (20) predicts $\Gamma \propto x^{-1/2}$, which follows from energy conservation since the wall does no work, whereas (2.6) would have Γ constant.

3. Non-planar shocks

No vorticity can be created by an exactly planar shock, yet the one-dimensional theory above may be adequate for estimating the vorticity produced by encounters between an ultra-relativistic shock and a density inhomogeneity. Lorentz contraction causes the lateral dimension of inhomogeneities viewed in the shock or post-shock frame to be larger by a factor $\Gamma \gg 1$ than the longitudinal ones, so that changes in speed and pressure are impressed upon the immediately post-shock flow before it 'notices' that the changes differ at other lateral positions. Thus it should usually be sufficient to evaluate the flow changes from the one-dimensional theory, and then take lateral derivatives to evaluate the resulting vorticity.

Nevertheless, it is worthwhile to extend the ultra-relativistic version of Whitham's theory to non-planar shocks for several reasons:

(i) in order to study the stability of the shock;

(ii) in order to compare with exact spherical and cylindrical self-similar solutions, and with numerical tests such as refraction around an (oblique) corner;

(iii) because the extension is not difficult.

The idea of Whitham's non-planar extension is to insert a factor representing changes in shock area into the conservative form of the fluid equations. Thus let x^1 be a coordinate measuring arclength along the shock normal, and x^2 and x^3 be coordinates in the shock surface defined in such a way that a point moving along the shock normal maintains constant (x^2, x^3). The equations of motion are taken to be

$$T^{00}{}_{,0} + A^{-1}(A\,T^{01})_{,1} = 0, \qquad T^{10}{}_{,0} + A^{-1}(A\,T^{11})_{,1} = 0. \tag{3.1}$$

Here A is the two-dimensional Jacobian relating the area of an element of the shock surface to its initial area. More precisely, since the characteristic equation does not hold across the shock, A represents the cross-sectional area of a bundle of streamlines immediately behind the shock; since the pre-shock medium is assumed to be at rest, the flow behind the shock is normal to it.

Equations (3.1) are not fully equivalent to $T^{\mu\nu}{}_{;\nu} = 0$: for $\mu = 1$, they do not contain the part of the covariant derivative associated with turning of the shock normal. They do however represent the divergence or convergence of the normals, which leads to area change and strengthens or weakens the shock. Because of the terms involving A, the quantities $R_\pm$ are no longer invariant along their respective characteristics $C_\pm$. The equation for R_+ then becomes

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)_+ (\zeta + \phi) = -\frac{\sinh\phi}{\sqrt{3}\,\sinh\phi + \cosh\phi}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)_+ \ln A$$

$$\to -\frac{1}{\sqrt{3}+1}\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)_+ \ln A\,, \tag{3.2}$$

where $(\mathrm{d}/\mathrm{d}s)_+ \equiv \partial_1 + (v_+)^{-1}\partial_0$ is the derivative along the C_+ characteristic, and $v_+ \equiv \tanh[\phi + \tanh^{-1}(1/\sqrt{3})]$ is the characteristic velocity. The final form of (3.2) is in the pre-shock rest frame where $\tanh\phi \approx 1$ with corrections $O(\Gamma^{-2}) \ll 1$.

The jump conditions (2.3) are unchanged. Inserting these into (3.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi + \lambda \ln\rho_0 + \mu \ln A\right) \approx 0, \tag{3.3a}$$

$$\lambda \equiv \sqrt{3} - \tfrac{3}{2}, \qquad \mu \equiv 3\sqrt{3} - 5. \tag{3.3b}$$

Following Whitham, we have made the approximation that the characteristic equation applies on the shock, although its propagation speed (rapidity $\Phi = \phi + \tanh^{-1}(1/3)$) is not quite the same as that of the characteristic ($\tanh^{-1} v_+ = \phi + \tanh^{-1}(1/\sqrt{3})$). Consistent with this approximation, the derivative $\mathrm{d}/\mathrm{d}s$ in (3.3) is taken to be the derivative with respect to arclength along the shock normal. Equation (3.3) predicts that the shock decelerates locally where its area increases, and accelerates where the area decreases. This will tend to stabilize corrugations in the shock front.

Equations (3.3) need to be supplemented by a vector equation for the shock normals. Introduce a function $\tau(x, y, z)$ such that the locus of the shock in Minkowski coordinates is described by $t = \tau(x, y, z)$. The normal to the shock is then $\boldsymbol{n} = \nabla\tau/|\nabla\tau|$, and its 3-velocity is $\boldsymbol{V} = \boldsymbol{n}/|\nabla\tau|$. The area function A of the shock satisfies

$$\nabla\boldsymbol{\cdot}\left(\frac{\boldsymbol{n}}{A}\right) = 0. \tag{3.4}$$

This is a purely geometrical, rather than dynamical, statement. Whitham (1974) demonstrates it by applying Gauss's Law to a 'flux tube' whose sides are made up of integral curves of $\boldsymbol{n}$, and whose ends are elements of the shock surface at different times.

3.1. *Comparison with non-planar self-similar solutions*

Sari (2006)'s equation (26) for Type II solutions is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}r}\ln\Gamma = \alpha(5 - 3\sqrt{3}) \;-\left(\sqrt{3} - \tfrac{3}{2}\right)\frac{\mathrm{d}}{\mathrm{d}r}\ln\rho,$$

$\rho \propto r^{-k}$ being the pre-shock density. Here $\alpha = 0, 1, 2$ for planar, cylindrical, and spherical shocks, respectively. Since the area factor should scale as $A \propto r^\alpha$ in these three geometries, equation (3.3) predicts Sari's result perfectly. Again, GSD does not work for the Type I solutions, which include those of Blandford & McKee (1976): for example, a spherical ultra-relativistic blast wave encountering a uniform external medium has $\Gamma \propto r^{-3/2}$ rather than Γ = constant as predicted by (3.3), because the self-similar behaviour is global and energy is conserved.

In these self-similar solutions, dimensionality plays a limited role since individual shock normals are constant. Whitham (1974) discusses applications of the non-relativistic theory to non-self-similar and truly multidimensional problems such as refraction of shocks around corners and obstacles. Here we can expect (3.3) and (3.4) not to be exact since, as noted above, they do not incorporate the full covariant derivatives in the equations of motion, and since the streamlines behind the shock are not perfectly straight.

3.2. *Detailed treatment of initially planar shocks in two dimensions*

These details illustrate effects that involve changes in the shock normal, including the transverse propagation of disturbances along the shock front.

We take z to be the ignorable coordinate. Following Whitham again, let ψ be the angle between the normal and the x-axis, so that $\partial\tau/\partial x = |\nabla\tau|\cos\psi = V^{-1}\cos\psi$ and $\partial\tau/\partial y = V^{-1}\sin\psi$. Equation (3.4) can then be rephrased as the two first-order equations

$$\frac{\partial}{\partial x}\left(V^{-1}\sin\psi\right) - \frac{\partial}{\partial y}\left(V^{-1}\cos\psi\right) = 0\,, \tag{3.5a}$$

$$\frac{\partial}{\partial x}\left(A^{-1}\cos\psi\right) + \frac{\partial}{\partial y}\left(A^{-1}\sin\psi\right) = 0\,, \tag{3.5b}$$

of which the first is simply the statement that $\partial^2\tau/\partial y\partial x = \partial^2\tau/\partial x\partial y$. Together with (3.3), equations (3.5) form a hyperbolic system. This is especially clear in the paraxial approximation where $\psi \sim O(\Gamma^{-2})$. To this order, we may then replace $\sin\psi \to \psi$, $\cos\psi \to 1$, and $V \to 1-(2\Gamma^2)^{-1}$, so that (3.5) become

$$\frac{\partial\psi}{\partial x} + \frac{1}{2}\frac{\partial}{\partial y}\left(\psi^2 - \Gamma^{-2}\right) = 0, \tag{3.6a}$$

$$\frac{\partial}{\partial x}A^{-1} + \frac{\partial}{\partial y}\left(A^{-1}\psi\right) = 0. \tag{3.6b}$$

With our ordering, the term in ψ^2 is of higher order than the others – it results from taking $\cos\psi = 1-\psi^2/2$ rather than unity in (3.6*a*) – but it does no harm and in fact makes the characteristic velocities work out more neatly.

If the shock is initially planar and the pre-shock density initially uniform, then (3.3) implies that $\Phi + \lambda\ln\rho_0 + \mu\ln A$ is constant throughout the flow. Thus Γ in (3.6*a*) is to be regarded as a function of A and (x, y), given by

$$\Gamma(A, x, y) = \bar{\Gamma}\bar{\rho}^{\lambda}\bar{A}^{\mu}\rho_0^{-\lambda}A^{-\mu}, \tag{3.7}$$

in which the barred quantities are constants pertaining to the initially uniform medium and planar shock, and $\rho_0(x, y)$ is a prescribed function. The characteristic velocities of the system (3.6)–(3.7) are

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\pm} = \psi \pm \mu^{1/2}\Gamma^{-1}. \tag{3.8}$$

The factor of Γ^{-1} is easy to interpret as a consequence of relativistic beaming. A disturbance propagating at the speed of light along an otherwise planar shock front would have transverse velocity $\mathrm{d}y/\mathrm{d}t = \pm\Gamma^{-1}$ in the pre-shock rest frame. To leading order in Γ^{-1}, we may replace $\mathrm{d}t$ with $\mathrm{d}x$ in this expression, so with $\mu^{1/2} \approx 0.4429$, the characteristics (3.8) are subluminal.

Since (3.5) and (3.6) are in conservation form, we may use them to study discontinuities in the shock front itself:'shock shocks'. For a shock shock propagating at slope $U \equiv (\mathrm{d}y/\mathrm{d}x)_{ss}$, the jump conditions implied by (3.6) are

$$\left[-2U\psi + \psi^2 - \Gamma^{-2}\right] = 0, \tag{3.9a}$$

$$\left[A^{-1}(\psi - U)\right] = 0, \tag{3.9b}$$

where $[Q]$ denotes the discontinuity in quantity Q across the shock shock. Let us assume a homogenous pre-shock medium, $\rho_0 = \bar{\rho} =$ constant, $\Gamma =$ constant. Then if $\psi = 0$ and $A = 1$ ahead of the shock shock, the post-shock-shock quantities (ψ', A') satisfy

$$(U\Gamma)^2 = \frac{(A')^{2\mu} - 1}{(A')^2 - 1}, \qquad \psi' = U(1 - A'). \tag{3.10}$$

Thus in the limit $A' \to 0$, we have $\psi' = U = 1/\Gamma$, and it follows from (3.7) that $\Gamma' \to \infty$. In the opposite limit $A' \gg 1$ – but still $A' \ll \Gamma^{1/\mu}$ so that $\Gamma' \gg 1$ (else the ultra-relativisitic approximation would not apply) – we have $\psi' = -(A')^{\mu}/\Gamma = -1/\Gamma'$ and $U = -\psi'/\Gamma'$.

Finally, because gamma-ray-burst shocks are believed to emanate from effectively point-like explosions, it is of interest to consider a nearly spherical rather than planar shock. This case is effectively two-dimensional if the perturbations are axisymmetric. We take polar coordinates (r, θ, ϕ) such that $\theta = 0, \pi$ is the axis of symmetry and define ψ to be the angle between the normal and radial directions, i.e. $\boldsymbol{n} \cdot \boldsymbol{r} = \cos\psi$. The analogues of (3.5) then become

$$\frac{\partial}{\partial r}\left(\frac{r\sin\psi}{V}\right) - \frac{\partial}{\partial\theta}\left(\frac{\cos\psi}{V}\right) = 0, \tag{3.11a}$$

$$\frac{\partial}{\partial r}\left(\frac{r^2\cos\psi}{A}\right) + \frac{r}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta\sin\psi}{A}\right) = 0, \tag{3.11b}$$

and for $\psi \ll 1$, $\Gamma \gg 1$, (3.6) become

$$\frac{\partial}{\partial r}(r\psi) + \frac{1}{2}\frac{\partial}{\partial\theta}\left(\psi^2 - \Gamma^{-2}\right) = 0, \tag{3.12a}$$

$$\frac{\partial}{\partial r}\left(\frac{r^2}{A}\right) + \frac{r}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\psi\sin\theta}{A}\right) = 0, \tag{3.12b}$$

while (3.7) is unchanged.

4. Relativistic vorticity

The discussion in this section is independent of the approximations of GSD, although the ultra-relativistic equation of state $P = \rho/3$ figures prominently.

As explained above, we are motivated by the need to explain the amplification of magnetic field behind the shocks associated with gamma-ray bursts, and by the possible role of turbulence in this amplification. Therefore, it may be worthwhile to record our assumptions about the relation of post-shock vorticity to the magnetic field.

It follows from the induction equation of ideal magnetohydrodynamics,

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{4.1}$$

that magnetic energy increases according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{B}\cdot\boldsymbol{B}\,\mathrm{d}^3\boldsymbol{x} = \int B_i B_j \partial_i v_j \,\mathrm{d}^3\boldsymbol{x}\,,$$

which involves the instantaneous shear $(\partial_i v_j + \partial_j v_i - \frac{2}{3}\nabla\cdot\boldsymbol{v})$ and convergence $(\nabla\cdot\boldsymbol{v})$ of the velocity field rather than the vorticity, which is its curl. Nevertheless, vorticity is important to secular amplification of the field by localized disturbances. In an ideal fluid, a localized non-vortical disturbance evolves into sound waves, whose oscillations produce only transitory changes in magnetic energy, and which propagate away from their source. Energy in vortical motions, however, remains localized, and the shear between neighbouring eddies is expected to amplify the field exponentially on their turnover time. For a fuller discussion see, for example, Kulsrud (2003).

4.1. *Vorticity and circulation*

Non-relativistically, the vorticity $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}$, where $\boldsymbol{v}$ is the fluid 3-velocity. In a compressible but isentropic fluid without shocks, Kelvin's Circulation theorem is

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_C \boldsymbol{v}\cdot\mathbf{d}\boldsymbol{l} = 0, \tag{4.2}$$

where C is closed contour advected by the flow.

The generalization of $\boldsymbol{\omega}$ and (4.2) to relativistic flow is not entirely straightforward (Eshraghi 2003). Let $\boldsymbol{U} = (U^0, U^1, U^2, U^3)$ be the 4-velocity of the fluid, so that $\eta_{\mu\nu}U^\mu U^\nu = -1$. In terms of the local rest-frame energy density ρ and pressure P, the energy-momentum tensor is

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu + \eta^{\mu\nu}P. \tag{4.3}$$

The equations of motion

$$T^{\mu\nu}{}_{,\nu} = 0 \tag{4.4}$$

must be supplemented by an equation of state. Normally this involves two independent thermodynamic variables, e.g. $P = P(\rho, T)$, $P(\rho, N)$, or $P(N, S)$, where T is the rest-frame temperature, N is the proper number density of conserved particles, and S is the entropy per particle. An essential feature of ultra-relativistic shocks is that the post-shock particles are highly relativistic in the fluid rest frame, so that $P = \rho/3$ (this assumes that the stress is isotropic, which is not at all obvious in astrophysical applications where the plasma is collisionless.) To the extent that the fluid is ideal (equation (4.3)), entropy and temperature gradients then have no effect on the flow – except at shocks, but even there they do not have to be addressed explicitly.

To illustrate this point, we consider a general equation of state in which entropy does influence the dynamics. The conservation of particle number is expressed by

$$(NU^\mu)_{,\mu} = 0. \tag{4.5}$$

The vorticity tensor is

$$\Omega_{\mu\nu} \equiv -H_{\mu,\nu} + H_{\nu,\mu}\,, \tag{4.6}$$

in terms of the relativistic enthalpy h and its associated current H^μ:

$$h \equiv \frac{\rho + P}{N}, \qquad H^\mu \equiv h U^\mu . \tag{4.7}$$

With the First Law of Thermodynamics $\mathrm{d}(\rho/N) = T\mathrm{d}S - P\mathrm{d}(1/N)$ in the form $N^{-1}\mathrm{d}P = \mathrm{d}h - T\mathrm{d}S$, (4.4) can be rewritten as

$$U^\nu H_{\mu,\nu} = T S_{,\mu} - h_{,\mu} . \tag{4.8}$$

The vorticity equation then follows from the curl of this, namely

$$\begin{aligned} -(U^\nu H_{\alpha,\nu})_{,\beta} + (U^\nu H_{\beta,\nu})_{,\alpha} &= U^\nu \Omega_{\alpha\beta,\nu} + U^\nu{}_{,\beta}\Omega_{\alpha\nu} + U^\nu{}_{,\alpha}\Omega_{\nu\beta} \\ &= T_{,\alpha}S_{,\beta} - T_{,\beta}S_{,\alpha} . \end{aligned} \tag{4.9}$$

The last line above vanishes if the entropy is uniform, $S_{,\mu} = 0$, or more generally if there is only one independent thermodynamic quantity so that $S = S(T)$. The intermediate expression is the Lie derivative $\mathscr{L}_U$ of the vorticity considered as a 2-form, $\boldsymbol{\Omega} \equiv \mathbf{d}\boldsymbol{H}$, in which $\mathbf{d}\boldsymbol{H} = H_\mu \mathbf{d}\boldsymbol{x}^\mu$ is the one-form embodiement of the enthalpy current and $\mathbf{d}\boldsymbol{H} = H_{\mu,\nu}\mathbf{d}\boldsymbol{x}^\nu \wedge \mathbf{d}\boldsymbol{x}^\mu$ is its exterior derivative. Thus the vorticity tensor is 'conserved' by isentropic flow in the sense that $\mathscr{L}_U\boldsymbol{\Omega} = 0$, and this can be shown to be equivalent to a circulation theorem. It would seem more natural to define the vorticity directly in terms of the 4-velocity, i.e. as $\mathbf{d}\boldsymbol{U}$, but the latter 'vorticity' is conserved only under conditions that are too restrictive for our purposes, such as pressureless flow or uniform cosmological expansion.

In non-relativistic fluid mechanics, we are accustomed to thinking of vorticity as a vector field rather than an antisymmetric tensor. Fortunately, vorticity can be expressed by a 3-vector even in relativity, at least in isentropic flow, because $\Omega_{\alpha\beta}$ then has only three independent components. One can show this by using (4.6) and

$$-h_{,\mu} = (H_\nu U^\nu)_{,\mu} = H_{\nu,\mu}U^\nu + H_\nu U^\nu{}_{,\mu} = H_{\nu,\mu}U^\nu + h(U_\nu U^\nu)_{,\mu} = H_{\nu,\mu}U^\nu ,$$

to rewrite (4.8) as

$$U^\nu \Omega_{\mu\nu} = T S_{,\mu} . \tag{4.10}$$

Therefore if the fluid is isentropic, then in the local fluid rest frame where $U^\nu \to \delta^\nu_0$, the 'electric' components $\Omega_{i0} = -\Omega_{0i}$ of the vorticity vanish, and only the three 'magnetic' components $\Omega_{ij} = -\Omega_{ji}$ survive. Let $\boldsymbol{\omega}$ be the three-vector field with components $\omega_i = \epsilon_{ijk}\Omega_{jk}/2$ in an arbitrary inertial frame (not necessarily coinciding with the local fluid rest frame). Using (4.10) with $S_{,\mu} = 0$ to eliminate the components Ω_{i0} in a general reference frame from the identity

$$\Omega_{\alpha\beta,\gamma} + \Omega_{\beta\gamma,\alpha} + \Omega_{\gamma\alpha,\beta} = 0,$$

which is the tensorial expression of $\mathbf{dd}\boldsymbol{H} = 0$, one obtains for isentropic conditions,

$$\partial_t \omega - \nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) = 0, \tag{4.11}$$

where $\boldsymbol{v}$ is the 3-velocity, $v^i = U^i/U^0$. This is formally identical to the non-relativistic vorticity equation of an isentropic fluid, except that $\boldsymbol{\omega}$ is $\nabla \times \boldsymbol{H}$ rather than $\nabla \times \boldsymbol{v}$. The derivation just given, which parallels that of the induction equation (4.1), shows that (4.11), like the ideal induction equation, is relativistically covariant even though it is formulated in terms of three vectors

Now we specialize to the ultra-relativistic equation of state, $P = \rho/3$. The true entropy per particle is $S = k_{\mathrm{B}} \ln(P^{3/4}/N) + \text{constant}$: the argument of the logarithm is proportional to the volume of phase space per particle in a non-degenerate relativistic ideal gas, which is probably the relevant limit for gamma-ray bursts (Landau & Lifshitz 1980). This entropy will not be uniform behind a shock that has passed over density inhomogeneities. However, we may define an ersatz number density $\tilde{N} \propto P^{3/4}$ at some initial time, so that $P/\tilde{N}^{4/3}$ is spatially uniform. If we demand that $\tilde{N}$ evolve according to (4.5) with $\tilde{N}$ replacing N, then $P/\tilde{N}^{4/3}$ will remain uniform in smooth parts of the flow, though not across shocks. This follows because $U_\mu T^{\mu\nu}{}_{,\nu} = -3U^\nu P_{,\nu} - 4PU^\mu{}_{,\mu} = 0$, whence $U^\nu(P/\tilde{N}^{4/3})_{,\nu} = 0$.

We do not have to deal with $\tilde{N}$ directly. Since $\tilde{h} \equiv (\rho + P)/\tilde{N} = 4P^{1/4}$, we can simply redefine the enthalpy current as

$$H^\mu = P^{1/4}U^\mu \qquad \text{(when } P = \rho/3 \text{ only).} \tag{4.12}$$

Then (4.8) becomes

$$U^\nu H_{\mu,\nu} = -\left(P^{1/4}\right)_{,\mu}, \qquad P = \left(H^\mu H_\mu\right)^2. \tag{4.13}$$

The vorticity defined in terms of this $\boldsymbol{H}$ via (4.6) is conserved in the sense that the right-hand side of (4.9) vanishes in all smooth parts of the flow, even after shocks. In particular, if $\boldsymbol{\Omega} = 0$ initially then it remains zero as long as the flow remains smooth. The jump conditions that follow from integrating (4.4) across shocks are not equivalent to the corresponding integral of (4.9), however, and so this vorticity can be created at shocks.

5. A numerical test of GSD

Whereas Johnson & McKee (1971) have shown that (2.5)–(2.6) hold for any decreasing pre-shock density profile, provided that the post-shock flow is uniform far downstream, we will now demonstrate that these relations can be accurate for densities that increase quite sharply in the direction of shock propagation.

A special case amenable to semi-analytic treatment is that of a stepwise increase, because it can be reduced to a relativistic Riemann problem. Consider a planar shock propagating towards positive x into a pressureless medium whose density $\rho_0(x)$ has the uniform values $\bar{\rho}$ and $\rho_R = f\bar{\rho}$ at $x < 0$ and $x > 0$, respectively, where $f > 1$. Let $t = 0$ at the instant when the shock arrives at $x = 0$. Before $t = 0$ there is but one shock, with rapidity $\bar{\Phi}$ measured in the pre-shock rest frame, and the shock separates two uniform states related by the jump conditions discussed in §2. The post-shock fluid rapidity and density are $\phi_L \approx \bar{\Phi} - \ln\sqrt{2}$ and $\rho_L \approx 2\bar{\rho}\cosh^2\bar{\Phi}$. After $t = 0$, there are two shocks: a forward shock with rapidity Φ_+ advancing into the new right-hand state, which has density ρ_R, pressure $P_R = 0$, and rapidity $\phi_R = 0$; and a reverse shock with rapidity Φ_- confronting the left-hand state, which has the same properties $(\rho, P, \phi) = (\rho_L, \rho_L/3, \phi_L)$ as those of the post-shock state at $t < 0$. Between the two shocks an intermediate state has come into existence, with properties $(\rho, P, \phi) = (\rho_I, \rho_I/3, \phi_I)$. The intermediate state is further divided into left- and right-hand parts separated by a contact discontinuity across which the pressure is continuous but the number density N is not; because of the ultra-relativistic equation of state, however, the contact discontinuity has no effect on the energy density and does not need to be taken into account when solving for $\Phi_\pm$. Hence the present problem is in fact easier than the general Riemann problem for an ideal non-relativistic gas. The following six equations determine the six unknowns

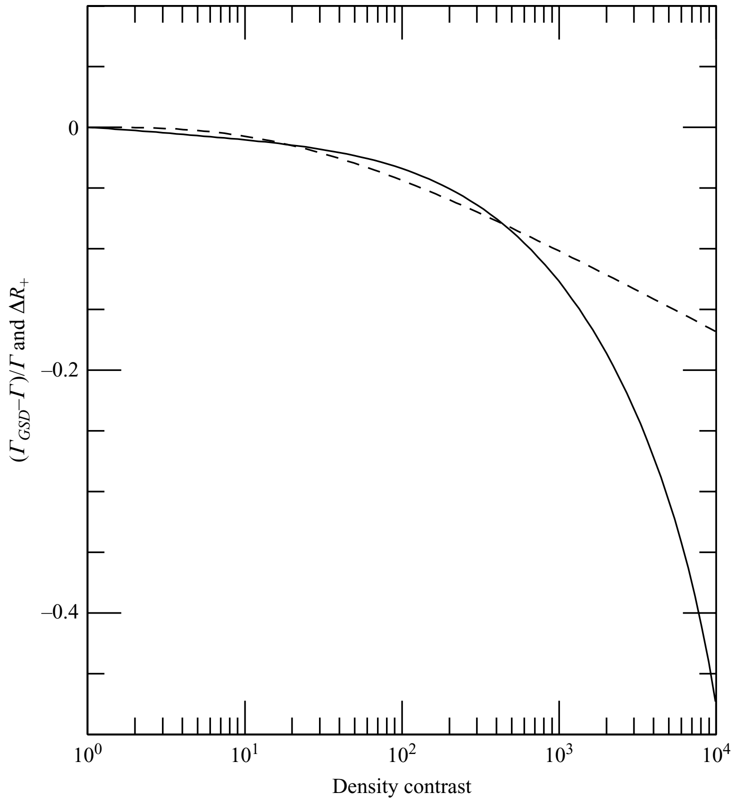

FIGURE 1. A test of GSD. A shock of initial Lorentz factor $\bar{\Gamma} = 10$ encounters a discontinuous increase in pre-shock density by the factor on the abscissa. The solid curve is the fractional error in Lorentz factor of the forward shock predicted by (2.6). The dashed curve is the change in the forward-going Riemann invariant (2.4) across the reverse shock.

$(\Phi_+, \Phi_-, \phi_L, \phi_I, \rho_L, \rho_I)$ in terms of the quantities $(\bar{\Phi}, \bar{\rho}, f)$ that are given:

$$\tfrac{4}{3}\rho_I \sinh[2(\Phi_+ - \phi_I)] = f\bar{\rho}\sinh 2\Phi_+, \tag{5.1a}$$

$$\tfrac{1}{3}\rho_I[4\sinh^2(\Phi_+ - \phi_I) + 1] = f\bar{\rho}\sinh^2\Phi_+, \tag{5.1b}$$

$$\rho_I \sinh[2(\Phi_- - \phi_I)] = \rho_L \sinh[2(\Phi_- - \phi_L)], \tag{5.1c}$$

$$\rho_I[4\sinh^2(\Phi_- - \phi_I) + 1] = \rho_L[4\sinh^2(\Phi_- - \phi_L) + 1], \tag{5.1d}$$

$$\tfrac{4}{3}\rho_L \sinh[2(\bar{\Phi} - \phi_L)] = \bar{\rho}\sinh 2\bar{\Phi}, \tag{5.1e}$$

$$\tfrac{1}{3}\rho_L[4\sinh^2(\bar{\Phi} - \phi_L) + 1] = \bar{\rho}\sinh^2\bar{\Phi}. \tag{5.1f}$$

Equations (5.1*a,b*) express the continuity of T^{01} and T^{11} at the forward shock; (5.1*c, d*), at the reverse shock; and (5.1*e, f*), at the shock that exists before $t = 0$. These equations are exact for the idealized problem described. In particular, the approximations $\bar{\Phi} \gg 1$ and $\phi_L \approx \bar{\Phi} - \ln\sqrt{2}$ have not been assumed. As usual, at each shock there are two solutions for the post-shock state in terms of the pre-shock quantities and the shock speed; we select the solutions that are subsonic in the shock frame.

Results for $\bar{\Gamma} = 10$ are shown in figure 1. The GSD approximation (2.6) predicts $\Gamma = f^{-\lambda}\bar{\Gamma}$ for the forward shock. In this test problem, the GSD prediction errs by no more than 10% for density contrasts up to $f \approx 600$. Recall that GSD is based on the approximation that the Riemann invariant on the C_+ characteristics i.e. those approaching the shock from the postshock side) has the same uniform value that it has far downstream in the post-shock flow. This approximation cannot be exact when there are reverse shocks, but the figure shows that the reverse shock changes R_+ little unless the density contrast is very large. Since (2.6) would predict $\Gamma < 1$ for $f > \bar{\Gamma}^{1/\lambda}$, self-consistency alone demands that GSD should fail above $f = 10^{4.3} \approx 2 \times 10^4$ in this case; remarkably, it does not fail much earlier.

In short, we find that the ultra-relativistic GSD approximation works surprisingly well even for very large density contrasts, both positive and negative. We have also developed an ultra-relativistic conservation law for vorticity. In a sequel to this work (Sironi & Goodman 2007), we combine these results to estimate the turbulence and magnetic-field amplification that may occur behind an ultra-relativistic gamma-ray-burst shock if it propagates into a suitable inhomogeneous medium.

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